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AN ASYMPTOTIC SOLUTION TO THE MEMBRANE EQUATIONS FOR SHELLS OF REVOLUTION OF NEGATIVE GAUSSIAN CURVATURE

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Abstract—This paper examines the membrane theory of the equilibrium of an elastic shell of revolution of negative Gaussian curvature, with a free edge and subject to a normal loading $P_n(\phi) \cos n\theta$, where θ is the meridional angle, ϕ is an angular parameter along meridian lines and n is an integer. The asymptotic dependence upon n, as $n \to \infty$, of the stress resultants at a fixed point of the shell is determined analytically to provide a basis of comparison with the full bending theory. The particular case of the single sheet hyperboloid is examined in detail.

1. INTRODUCTION

ANALYSIS of the stresses in an elastic shell of revolution has important engineering applications, a particular example being the construction of concrete cooling towers in the shape of single sheet hyperboloids. It is well known [1] that the equilibrium of an elastic shell is described by an eighth order system of linear partial differential equations but it is a considerable task to solve boundary value problems for this system. Consequently approximate methods are often used to obtain solutions for engineering design purposes. One of the most common approximations—the membrane approximation—ignores the couple resultants altogether and by so doing leads to a second order system of linear equations. The reduction in order, from eighth to second, necessarily results in the loss of boundary conditions but the hope is that, away from certain narrow "boundary layers" where bending effects are needed to satisfy the "lost" boundary conditions, the membrane description is adequate.

Recently solutions of the full eighth order system and the second order membrane system have been compared [2] for hyperboloids of revolution with one edge free and one edge clamped and subjected to normal wind loadings of the form

$$\sum_n P_n(\phi) \cos n\theta,$$

where θ is the meridional angle, ϕ is an angular parameter along meridian lines and n is an integer. The linearity of the equations and the homogeneity of the boundary conditions together imply that the state of stress is a superposition of the stresses due to each separate harmonic $P_n(\phi) \cos n\theta$ of the loading. It was found that for a normal loading $P_n(\phi) \cos n\theta$ and "small" values of n the bending and membrane theories are in good agreement but that for sufficiently large values of n the two solutions are markedly dissimilar and the membrane approximation cannot then be considered adequate. This conclusion is, perhaps, not

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surprising in view of the fact that the couple resultants, which might be expected to play an important rôle for large n, are neglected by the membrane theory. However, the precise asymptotic behaviours of the bending and membrane solutions for large n and the way in which they differ are not so obvious.

In this paper the behaviour of the second order membrane system is examined analytically for shells having a free edge. A normal loading $P_n(\phi)\cos n\theta$ is assumed and the asymptotic dependence of the solution upon the harmonic number n as $n \to \infty$ is determined firstly for a hyperboloid of revolution, which introduces certain simplifying features, and secondly for a general shell of revolution of negative Gaussian curvature. The asymptotic expansion derived for the hyperboloid is shown to give excellent agreement with the results of a direct numerical solution of the differential equations. The expansion confirms that, on the membrane theory, the stress resultants at a fixed point of the hyperboloid do not approach a limiting value as $n \to \infty$ but oscillate between two fixed values. In contrast the calculations in [2] indicate that on the bending theory the stress resultants approach the value zero as a limit. The analysis for the general shell shows that the oscillatory asymptotic behaviour of the membrane solution is not particular to the hyperboloid but characterizes the solution for any shell of revolution of negative Gaussian curvature.

2. FORMULATION OF THE PROBLEM

We take a system of rectangular Cartesian coordinates (x, y, z) and suppose the middle surface of the shell generated by revolving the arc

$$x = x(\phi) > 0, \qquad z = z(\phi), \qquad 0 < \phi_0 \leq \phi \leq \phi_1 < \pi \tag{2.1}$$

about the z-axis. The parameter ϕ is defined as follows: if $Q(\phi)$ is a point of the arc then ϕ is the angle which the normal to the arc at Q makes with the positive direction of the z-axis. It is assumed that

$$\mathrm{d}^2x/\mathrm{d}z^2 > 0 \tag{2.2}$$

for all ϕ in $\phi_0 \leq \phi \leq \phi_1$ so that the surface of revolution then has negative Gaussian curvature. The free edge of the shell can, without essential loss of generality, be taken to be the circle $\phi = \phi_0$.

The membrane equations of equilibrium for the stress resultants $N_\phi, N_\theta, N_{\phi\theta} = N_{\theta\phi}$ have been given by Timoshenko and Woinowsky-Krieger [3]. They are

$$\begin{aligned} &\frac{\partial}{\partial\phi}(r_2 \sin\phi N_\phi) + r_1\frac{\partial N_{\phi\theta}}{\partial\theta} - r_1\cos\phi N_\theta = 0 \\ &\frac{\partial}{\partial\phi}(r_2 \sin\phi N_{\phi\theta}) + r_1\frac{\partial N_\theta}{\partial\phi} + r_1\cos\phi N_{\phi\theta} = 0 \\ &r_2N_\phi + r_1N_\theta + r_1r_2P_n\cos n\theta = 0, \end{aligned} \tag{2.3}$$

where $r_1(\phi)$, $r_2(\phi)$ are the principal radii of curvature of the shell. The boundary conditions at the free edge are

$$N_\phi(\phi_0) = 0, \qquad N_{\phi\theta}(\phi_0) = 0. \tag{2.4}$$

It can be verified that the characteristic equation of the system of two first order equations for N_ϕ and $N_{\phi\theta}$, which are obtained by eliminating N_θ, is

$$r_1 r_2 \, d\phi^2 + r_2^2 \sin^2\phi \, d\theta^2 = 0. \tag{2.5}$$

Moreover, the assumption (2.2) implies that the shell has negative Gaussian curvature, that is to say

$$r_1(\phi) r_2(\phi) < 0 \tag{2.6}$$

for all ϕ in $\phi_0 \leq \phi \leq \phi_1$. Consequently the system has real distinct characteristics and is hyperbolic. The form of the conditions (2.4) implies that we have to solve an initial value problem and the hyperbolic character of the equations then ensures that this problem is well posed.

To solve (2.3) we write

$$N_\phi = N_{\phi n}(\phi) \cos n\theta, \qquad N_{\phi\theta} = N_{\phi\theta n}(\phi) \sin n\theta, \qquad N_\theta = N_{\theta n}(\phi) \cos n\theta. \tag{2.7}$$

Substituting from (2.7) into (2.3) and eliminating $N_{\phi\theta n}$, $N_{\theta n}$ produces a second order ordinary differential equation for $N_{\phi n}$ of the form

$$\frac{d^2 N_{\phi n}}{d\phi^2} + f_1(\phi) \frac{dN_{\phi n}}{d\phi} + \left(\frac{-n^2 r_1 r_2}{r_2^2 \sin^2\phi} + f_2(\phi) \right) N_{\phi n} = \frac{n^2 r_1^2}{r_2 \sin^2\phi} P_n + f_3(\phi), \tag{2.8}$$

where f_1, f_2 are determined by r_1, r_2 alone and f_3 is determined by r_1, r_2 and the prescribed P_n. The conditions (2.4) become

$$N_{\phi n}(\phi_0) = 0, \qquad \frac{dN_{\phi n}}{d\phi}(\phi_0) = -r_1(\phi_0) \cot \phi_0 \, P_n(\phi_0). \tag{2.9}$$

Our objective is to determine the asymptotic behaviour of $N_{\phi n}(\phi)$ as $n \to \infty$ for sufficiently smooth $r_1(\phi)$, $r_2(\phi)$, $P_n(\phi)$ subject only to the condition (2.6).

3. THE HYPERBOLOID

For the particular case of the hyperboloid of revolution it is simpler to take advantage of a transformation described by Martin and Scriven [4] rather than use equations (2.8), (2.9) directly.

We consider the hyperboloid obtained by revolving the arc

$$x^2/a^2 - z^2/b^2 = 1, \qquad x \geq 0 \tag{3.1}$$

about the z axis. A convenient parametric representation of the hyperboloid is

$$x = a \sec t \cos \theta, \qquad y = a \sec t \sin \theta, \qquad z = -b \tan t, \tag{3.2}$$

where the parameter t is related to ϕ by the formulae

$$\alpha(t) \sin \phi = b \sec t, \qquad \alpha^2(t) \, d\phi/dt = -ab \sec t, \tag{3.3}$$

the function α being defined by

$$\alpha(t) = (a^2 \tan^2 t + b^2 \sec^2 t)^{\frac{1}{2}}. \tag{3.4}$$

Considered as functions of t, the principal radii of curvature are

$$r_1(t) = -\alpha^3(t)/ab, \qquad r_2(t) = a\alpha(t)/b. \tag{3.5}$$

On making the transformation

$$N_{\phi n}(t) = \frac{\alpha(t)}{a \sec^2 t} R_n(t). \tag{3.6}$$

we find that equation (2.8) reduces to the equation

$$\frac{d^2 R_n}{dt^2} + n^2 R_n = \frac{n^2 \alpha^2 \sec^2 t}{b} P_n - \frac{a^2}{b} \frac{d}{dt}\left(\frac{\sin t}{\cos^3 t} P_n\right) \tag{3.7}$$

and the conditions (2.9) to the conditions

$$R_n(t_0) = 0, \qquad \frac{dR_n(t_0)}{dt} = -\frac{a^2 \sin t_0}{b \cos^3 t_0} P_n(t_0) \tag{3.8}$$

where $\alpha(t_0) \sin \phi_0 = b \sec t_0$. The function R_n can be determined immediately from equations (3.7) and (3.8) in the form

$$\begin{aligned} R_n(t) = {} & \frac{\alpha^2(t) \sec^2 t}{b} P_n(t) - \frac{\alpha^2(t_0) \sec^2 t_0}{b} P_n(t_0) \cos n(t - t_0) \\ & - \left\{ \frac{a^2 \sin t_0}{b \cos^3 t_0} P_n(t_0) + \frac{d}{dt}\left(\frac{\alpha^2(t) \sec^2 t\, P_n(t)}{b}\right)\Bigg|_{t=t_0} \right\} \frac{1}{n} \sin n(t-t_0) + S_n(t) \end{aligned} \tag{3.9}$$

where $S_n(t)$ satisfies the equation

$$\begin{aligned} \frac{d^2 S_n}{dt^2} + n^2 S_n &= -\frac{a^2}{b}\frac{d}{dt}\left(\frac{\sin t}{\cos^3 t} P_n\right) - \frac{d^2}{dt^2}\left(\frac{\alpha^2(t) \sec^2 t\, P_n}{b}\right) \\ &= T(t), \text{ say}, \end{aligned} \tag{3.10}$$

together with the initial conditions

$$S_n(t_0) = 0, \qquad \frac{dS_n(t_0)}{dt} = 0. \tag{3.11}$$

The homogeneity of the initial conditions (3.11) now enables us to express the solution of (3.10) in the form

$$S_n(t) = \frac{1}{n} \int_{t_0}^{t} T(t') \sin n(t - t')\, dt'. \tag{3.12}$$

An asymptotic expansion for S_n, and hence for R_n, valid as $n \to \infty$ can be deduced by expanding the integral (3.12) by parts and using the Riemann–Lebesgue Lemma [5] namely, that provided the function g is integrable

$$\lim_{n \to \infty} \int_{t_0}^{t} g(t') \sin n\, t'\, dt' = \lim_{n \to \infty} \int_{t_0}^{t} g(t') \cos n\, t'\, dt' = 0.$$

In this way we find

$$
\begin{aligned}
S_n(t) = {} & \frac{1}{n^2}(T(t)-T(t_0)\cos n\,(t-t_0)) - \frac{1}{n^3}\frac{\mathrm{d}T(t_0)}{\mathrm{d}t}\sin n\,(t-t_0) \\
& -\frac{1}{n^4}\left(\frac{\mathrm{d}^2T(t)}{\mathrm{d}t^2} - \frac{\mathrm{d}^2T(t_0)}{\mathrm{d}t^2}\cos n\,(t-t_0)\right) + o\left(\frac{1}{n^4}\right).
\end{aligned}
\tag{3.13}
$$

Substituting this expression into equation (3.9) and thence into equation (3.6) gives an expression for $N_{\phi n}(t)$ valid as far as terms which are $o(1/n^4)$.

As a check this expansion has been evaluated for a particular cooling tower and the value compared with the direct numerical solution of equation (3.7). The relevant parameters are $a = 84$ ft, $b = 209{\cdot}661$ ft; the free edge is at $t = t_0 = -0{\cdot}2787265$ rad and the tower is under a loading for which $P_n(t) = 1$ lbf/ft^2. The expansion for $N_{\phi 100}(t)$, omitting terms which are $o(1/n^4)$, at $t = 0{\cdot}9098175$ rad gives the value 2252·15 lbf/ft whereas the value computed by a Runge–Kutta procedure is 2252·14 lbf/ft.

The essential behaviour of $N_{\phi n}$ is brought out by examining merely the leading terms in the expansion. We have, from (3.6) and (3.9),

$$
N_{\phi n}(t) \sim \frac{\alpha^3(t)}{ab}P_n(t) - \frac{\alpha(t)\alpha^2(t_0)}{ab}\frac{\sec^2 t_0}{\sec^2 t}P_n(t_0)\cos n\,(t-t_0). \tag{3.14}
$$

It is apparent that, for each fixed $t \neq t_0$, $N_{\phi n}(t)$ does not approach a limiting value as $n \to \infty$ but rather oscillates between two fixed values which depend on t and whose arithmetic mean is $\alpha^3(t)P_n(t)/ab = -r_1(t)P_n(t)$. On the other hand $N_{\phi n}$ calculated on the bending theory [2], for the same hyperboloid with the same normal loading $P_n(\phi)\cos n\theta$ and with one edge free and one clamped, appears to satisfy the relation

$$
N_{\phi n}(t) \to 0 \quad \text{as} \quad n \to \infty \tag{3.15}
$$

for each fixed t.

4. THE GENERAL SHELL OF REVOLUTION OF NEGATIVE GAUSSIAN CURVATURE

In this section we consider the general case when $N_{\phi n}$ is defined by equations (2.8), (2.9) and show how to obtain the appropriate asymptotic expansion. Since an examination of the leading term of the expansion suffices to show that the behaviour of $N_{\phi n}$ described in Section 3 is not peculiar to the hyperboloid but occurs in the general case, we derive here only the leading term. Subsequent terms in the expansion can be determined, if required, by a systematic procedure which is described but we do not produce them explicitly.

In view of the form of equation (2.8) and of the conditions (2.9) it is convenient to represent $N_{\phi n}$ in terms of a Green's function $G_n(\phi, \phi_0)$ defined as the solution of the homogeneous equation

$$
\frac{\mathrm{d}^2G_n}{\mathrm{d}\phi^2} + f_1(\phi)\frac{\mathrm{d}G_n}{\mathrm{d}\phi} + \left(\frac{-n^2r_1r_2}{r_2^2\sin^2\phi} + f_2(\phi)\right)G_n = 0 \tag{4.1}
$$

with initial values

$$G_n(\phi_0, \phi_0) = 0, \qquad \frac{dG_n}{d\phi}(\phi_0, \phi_0) = 1. \tag{4.2}$$

The representation is [6]

$$\begin{aligned} N_{\phi n}(\phi) = & -r_1(\phi_0)\cot\phi_0\, P_n(\phi_0)G_n(\phi, \phi_0) \\ & + \int_{\phi_0}^{\phi} G_n(\phi, \phi')\left(\frac{n^2 r_1^2(\phi')P_n(\phi')}{r_2(\phi')\sin^2\phi'} + f_3(\phi')\right) d\phi', \end{aligned} \tag{4.3}$$

where the first term is the solution to the homogeneous equation corresponding to (2.8) with initial values (2.9) and the second term is the solution to the inhomogeneous equation (2.8) with zero initial values. Once the asymptotic behaviour of G_n as $n \to \infty$ has been determined the asymptotic behaviour of $N_{\phi n}$ can be deduced from (4.3). The asymptotic behaviour of G_n will be found by using Liouville–Green or (WKB) techniques [7].

Since condition (2.6) holds we can define the real indefinite integral

$$s(\phi) = \int^{\phi} \frac{[-r_1(\phi')r_2(\phi')]^{\frac{1}{2}}}{r_2(\phi')\sin\phi'} d\phi' \tag{4.4}$$

and seek an asymptotic expansion for G_n of the form

$$\begin{aligned} G_n(\phi, \phi_0) \sim & \cos n[s(\phi)-s(\phi_0)] \sum_{r=0}^{\infty} \frac{A_r}{n^r}(\phi, \phi_0) \\ & + \sin n[s(\phi)-s(\phi_0)] \sum_{r=0}^{\infty} \frac{B_r}{n^r}(\phi, \phi_0). \end{aligned} \tag{4.5}$$

In order to determine the functions A_r, B_r we substitute from (4.5) into (4.1), equate to zero the coefficients of $\cos n[s(\phi)-s(\phi_0)]$, $\sin n[s(\phi)-s(\phi_0)]$ and then compare coefficients of inverse powers of n to find the equations

$$\frac{dA_0}{d\phi} + \frac{dF}{d\phi}A_0 = 0 \tag{4.6}$$

$$\frac{dA_r}{d\phi} + \frac{dF}{d\phi}A_r - L(B_{r-1}) = 0, \qquad r \geq 1 \tag{4.7}$$

$$\frac{dB_0}{d\phi} + \frac{dF}{d\phi}B_0 = 0 \tag{4.8}$$

$$\frac{dB_r}{d\phi} + \frac{dF}{d\phi}B_r + L(A_{r-1}) = 0, \qquad r \geq 1 \tag{4.9}$$

where the function F is the indefinite integral

$$F(\phi) = \frac{1}{2}\int^{\phi} f_1(\phi')\, d\phi' + \frac{1}{2}\log\frac{ds}{d\phi} \tag{4.10}$$

and L is the operator

$$L = \frac{1}{2\mathrm{d}s/\mathrm{d}\phi}\left(\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}+f_1\frac{\mathrm{d}}{\mathrm{d}\phi}+f_2\right). \tag{4.11}$$

The initial conditions at $\phi = \phi_0$ are found to be

$$A_r(\phi_0, \phi_0) = 0, \qquad r \geq 0 \tag{4.12}$$

$$B_0(\phi_0, \phi_0) = 0 \tag{4.13}$$

$$B_1(\phi_0, \phi_0) = \frac{1}{\mathrm{d}s(\phi_0)/\mathrm{d}\phi}\left(1-\frac{\mathrm{d}A_0}{\mathrm{d}\phi}(\phi_0, \phi_0)\right) \tag{4.14}$$

$$B_r(\phi_0, \phi_0) = \frac{\mathrm{d}A_{r-1}(\phi_0, \phi_0)/\mathrm{d}\phi}{\mathrm{d}s(\phi_0)/\mathrm{d}\phi}, \qquad r \geq 2. \tag{4.15}$$

Equations (4.6)–(4.9) and the initial values (4.12)–(4.15) enable us to determine the A_r, B_r by forward integration of first order ordinary differential equations. In fact (4.6), (4.7) and (4.12) imply

$$A_0(\phi, \phi_0) = A_1(\phi, \phi_0) = 0. \tag{4.16}$$

whilst (4.8) and (4.13) imply

$$B_0(\phi, \phi_0) = 0. \tag{4.17}$$

Also (4.9) and (4.14) give

$$B_1(\phi, \phi_0) = \frac{1}{\mathrm{d}s(\phi_0)/\mathrm{d}\phi}\mathrm{e}^{F(\phi_0)-F(\phi)}. \tag{4.18}$$

In general, for all $r \geq 2$, equations (4.7), (4.9) integrate to give

$$A_r(\phi, \phi_0) = \mathrm{e}^{-F(\phi)}\int_{\phi_0}^{\phi} \mathrm{e}^{F(\phi')}L[B_{r-1}(\phi')]\,\mathrm{d}\phi' \tag{4.19}$$

$$B_r(\phi, \phi_0) = -\frac{\mathrm{d}A_r(\phi_0, \phi_0)/\mathrm{d}\phi}{\mathrm{d}s(\phi_0)/\mathrm{d}\phi}\mathrm{e}^{F(\phi_0)-F(\phi)} - \mathrm{e}^{-F(\phi)}\int_{\phi_0}^{\phi} \mathrm{e}^{F(\phi')}L[A_{r-1}(\phi')]\,\mathrm{d}\phi' \tag{4.20}$$

and all the A_r, B_r can be found from these relations.

The leading term in the expansion of the Green's function is thus

$$G_n(\phi, \phi_0) \sim \frac{1}{\mathrm{d}s(\phi_0)/\mathrm{d}\phi}\,\mathrm{e}^{F(\phi_0)-F(\phi)}\frac{1}{n}\sin n\,[s(\phi)-s(\phi_0)],$$

which, by (4.4), is

$$G_n(\phi, \phi_0) \sim \frac{r_2(\phi_0)\sin\phi_0}{(-r_1(\phi_0)r_2(\phi_0))^{\frac{1}{2}}}\,\mathrm{e}^{F(\phi_0)-F(\phi)}\frac{1}{n}\sin n\,[s(\phi)-s(\phi_0)]. \tag{4.21}$$

A knowledge of the leading term allows us to determine the leading term in the asymptotic expansion of $N_{\phi n}$. We find from (4.3) that for large n

$$N_{\phi n} \sim n\,\mathrm{e}^{-F(\phi)}\int_{\phi_0}^{\phi} \frac{r_1^2(\phi')\,\mathrm{e}^{F(\phi')}P_n(\phi')}{[-r_1(\phi')r_2(\phi')]^{\frac{1}{2}}\sin\phi'}\sin n\,[s(\phi)-s(\phi')]\,\mathrm{d}\phi'. \tag{4.22}$$

On using (4.4) it follows that (4.22) can be written as

$$N_{\phi n} \sim e^{-F(\phi)} \int_{\phi_0}^{\phi} -r_1(\phi')\, e^{F(\phi')} P_n(\phi') \frac{d}{d\phi'} \{\cos n\, [s(\phi) - s(\phi')]\}\, d\phi', \tag{4.23}$$

and on integrating (4.23) by parts and using the Riemann–Lebesgue Lemma again we see that

$$N_{\phi n} \sim e^{-F(\phi)} \Big[-r_1(\phi')\, e^{F(\phi')} P_n(\phi') \cos n\, [s(\phi) - s(\phi')] \Big]_{\phi' = \phi_0}^{\phi' = \phi}.$$

Thus the leading term in the expansion for $N_{\phi n}$ is

$$N_{\phi n} \sim -r_1(\phi) P_n(\phi) + e^{F(\phi_0) - F(\phi)} r_1(\phi_0) P_n(\phi_0) \cos n\, [s(\phi) - s(\phi_0)]. \tag{4.24}$$

Once again, in this general case, the essential feature of the asymptotic expansion is that, for each fixed $\phi > \phi_0$, $N_{\phi n}(\phi)$ does not tend to a limit as $n \to \infty$ but instead its values oscillate about the mean value $-r_1(\phi) P_n(\phi)$.

It will be observed that the discussion of the hyperboloid presents simplifications when discussed in terms of the independent variable $t = t(\phi)$ in part because, to within an undetermined additive constant, $t(\phi) \equiv s(\phi)$. That this is so is seen by noting that equations (3.3), (3.5) and the definition (4.4) of $s(\phi)$ imply

$$\frac{ds}{d\phi} = \frac{[-r_1(\phi) r_2(\phi)]^{\frac{1}{2}}}{r_2(\phi) \sin\ \phi} = \frac{dt}{d\phi}.$$

Without much labour the leading term (3.14) in the special case can be identified with the leading term (4.24) in the general case.

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Résumé—Ce rapport examine la théorie de membrane de l'équilibre d'une couche élastique de révolution à courbure de Gauss négative, ayant un bord libre et sujette à un chargement normal $P_n(\phi)$ cos $n\theta$, où θ est l'angle méridional, ϕ est un paramètre angulaire le long des lignes méridiennes et n est un entier. La dépendance asymptotique sur n, comme $n \to \infty$ des résultantes de contrainte à un point fixe de la couche est déterminée analytiquement pour former une base de comparaison avec la théorie complète de flexion. Le cas particulier d'un hyperboloide à feuille simple est examiné en détails.

Zusammenfassung—Diese Arbeit untersucht die Membranentheorie des Gleichgewichtes elastischer Drehungsschalen mit negativer Gauss'scher Krümmung, mit freiem Ende und unter Normal-last $P_n(\phi)\cos n\theta$; wobei θ den Meriodonalwinkel, ϕ einen Winkelparameter entlang den Meridianlinien und n eine ganze Zahl darstellen. Die asymptotische Abhängigkeit von n, in der Form $n \to \infty$, des Spannungslinien in einem festen Punkt der Schale wird analytisch bestimmt, um als Vergleich mit der vollen Biegungstheorie zu dienen. Der besondere Fall eines Einzelplatten-Hyperboloides wird genau untersucht.

Абстракт—Эта статья исследует мембранную теорию равновесия эластической оболочки оборота отрицательной гаусовской кривизны со свободным краем и при условии нормального нагружения $P_n(\phi)\cos n\theta$, где θ—меридиональный угол, ϕ—угловой параметр вдоль линий меридиана и n—целое число. Асимптотическая зависимость от n, так как $n \to \infty$, результантов напряжения у закреплённого пункта оболочки определяется аналитически, чтобы дать основание для сравнения с теорией полного изгибания. Детально рассматривается особый случай однополостного гиперболоида.